Journal of Statistical Physics, Vol. 30, No. 2, 1983

On Pearson's Random Walk and Some Statistical Properties of a Quasiperiodic Observable in a Simple Quantum Model

F. T. Hioe[1]

The dynamics of the atomic inversion of a quantum model of field–atom interaction is studied from a statistical point of view. We determine its mean motion and its partial recurrence frequencies. We employed the mathematical analysis used by Lagrange, Wintner, and Weyl in their pioneering studies of the perturbed planetary motion and its connection with the studies of Pearson's random walks.

KEY WORDS: Atomic inversion; mean motion; partial recurrence.

A model Hamiltonian which has been frequently used in the studies of the resonant interaction of radiation with matter[1] is

$$\hat{H} = \tfrac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right) \tag{1}$$

where the $\hat{\sigma}$'s are the usual 2×2 Pauli matrices, $\hat{a}$ and $\hat{a}^\dagger$ are the Bose operators for the quantized field mode, ω_0 the natural transition frequency of the atom, ω the mode frequency, and λ the field–atom coupling parameter. The model is called the Jaynes–Cummings model[2] in quantum optics and is essentially identical to the spin-phonon model of NMR. It is one of the very few known quantum mechanical models that is exactly soluble for arbitrarily large coupling constants. Its dynamics, however, have not been fully investigated until very recently.[3,4]

The exact solution for the atomic inversion $w(t) \equiv \langle\sigma_z(t)\rangle$ is known to be given by

$$w(t) = m\sum_{k=0}^{\infty} p_k\left\{\frac{\Delta^2}{\Omega_k^2(m)} + \left[1 - \frac{\Delta^2}{\Omega_k^2(m)}\right]\cos\Omega_k(m)t\right\} \tag{2}$$

Presented at the Symposium on Random Walks, Gaithersburg, MD, June 1982.
Research supported in part by the U.S. Department of Energy.

[1] Department of Physics, St. John Fisher College, Rochester, New York 14618.

where $m = +1$ or -1 represents the initial state of the atom, p_k represents the initial photon distribution which, for the cases (coherent and chaotic) which are interested in, are given, respectively, by

$$p_k = \begin{cases} (\bar{n})^k e^{-\bar{n}}/k! & \text{coherent (Poissonian)} \quad (3a) \\ (\bar{n}+1)^{-1}\left[\bar{n}/(\bar{n}+1)\right]^k & \text{chaotic (thermal)} \quad (3b) \end{cases}$$

$\bar{n}$ being the mean photon number. The Δ in Eq. (2) denotes the field–atom frequency difference

$$\Delta = \omega_0 - \omega \tag{4}$$

and $\Omega_k(m)$ denotes

$$\Omega_k(m) = \left[\Delta^2 + 4\lambda^2\left(k + \frac{m+1}{2}\right)\right]^{1/2} \tag{5}$$

The physical meaning of $w(t)$ is that it represents on a scale between -1 and $+1$ the degree of excitation of a two-level system brought about by the interaction of the atom and the radiation field. The sum in Eq. (2) has no known closed form analytic expression.

The first significant property of $w(t)$ given by Eq. (2) when the initial photon distribution is coherent was discovered by Eberly, Narozhny, and Sanchez-Mondragon.[3] They noted the very interesting recurring "collapses" and "revivals" of the quantity $w(t)$ as shown in Fig. 1 (in the scaled time $\tau \equiv \lambda t/2\sqrt{\bar{n}}$ interval $0 \leqslant \tau \leqslant 4\pi$), and they were able to derive an approximate analytic expression for $w(t)$ that accurately represented

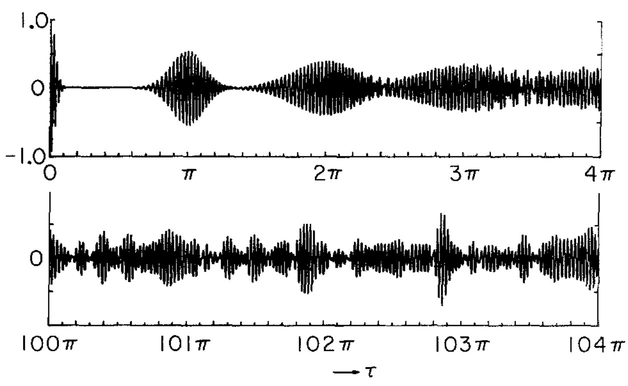

Fig. 1. Time histories of the atomic inversion $w(t)$ with the atom initially in the "down" state ($m = -1$) and the field initially in a coherent state with the average photon number $\bar{n} = 20$. The figure covers two widely separated time intervals of the same length $0 \leqslant \tau \leqslant 4\pi$ and $100\pi \leqslant \tau \leqslant 104\pi$, where the scaled time τ is defined by $\tau \equiv \lambda t/2\sqrt{\bar{n}}$.

these behaviors. On a longer time scale, however, as shown in the second part of Fig. 1, neighboring revivals begin to overlap, and the overlaps gradually include more distant revivals, causing the envelope of the atomic inversion signal to appear increasingly irregular. When the initial photon distribution is chaotic as given by Eq. (3b), no similar distinctive "collapses" or "revivals" behaviors were observed, and the quantity $w(t)$ appeared irregular from the beginning.

Our interest here concerns the properties of the quasiperiodic function $w(t)$ in the region after a long time, compared to the initial collapse and revival times, namely, in the region where the behavior of this function appears irregular. We ask what quantities should characterize these irregular behaviors? Is there a statistical distribution, and is there a meaningful mean frequency of oscillation? The purpose of this note is to summarize the results of our recent work[5] in this direction.

We were very fortunate that the mathematical analysis required for our study has been worked out for us by Lagrange, Wintner,[6] Weyl,[7] and others in their pioneering study of the perturbed planetary orbits, and that a beautiful exposition of these works has been given by Montroll.[8]

We first note that a generalized version of Eq. (2) represented by finite or infinite sums of the form

$$x(t) = \sum_{k=0}^{N} a_k \cos(\Omega_k t + \delta_k) \tag{6}$$

where all the a_k's are positive and the Ω_k's linearly independent, have been extensively studied first in connection with the perturbed planetary motion, and subsequently in statistical mechanics[9,10]; and the problem is known to be closely connected with the problem of the so-called Pearson's random walk[11] first studied by Rayleigh.[12] The linear independence of Ω_k's is strictly true in our case if Δ^2 in Eq. (5) is any (arbitrary) irrational number. We are interested in the case when Δ^2 is very small. Without loss of generality, we assume Δ^2 to be a small irrational number.

The Kronecker–Weyl theorem[13] can then be used to replace the time average in the characteristic function $f(\alpha)$ of $x(t)$ of Eq. (6) given by

$$f(\alpha) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \exp\left[i\alpha x(t) \right] dt \tag{7}$$

by the corresponding phase average and get

$$\begin{aligned} f(\alpha) &= \frac{1}{(2\pi)^{N+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \exp\left(i\alpha \sum_{k=0}^{N} a_k \cos\phi_k \right) d\phi_0 \ldots d\phi_N \\ &= \prod_{k=0}^{N} J_0(a_k \alpha) \end{aligned} \tag{8}$$

where $J_0(x)$ is the zeroth-order Bessel function. The probability density $P(x)$ of $x(t)$ is the Fourier transform of the characteristic function $f(\alpha)$ and is therefore given by

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{k=0}^{N} J_0(a_k\alpha)e^{-i\alpha x}\,d\alpha \tag{9}$$

which is the first useful quantity for characterizing the distribution of $x(t)$.

The second useful quantity for characterizing $x(t)$ is its "partial recurrence," or the average frequency with which $x(t)$ achieves particular value q. One defines $L(q)$, the average frequency $x(t) = q$ by

$$L(q) = \lim_{T\to\infty}\frac{1}{T}N_T(q) \tag{10}$$

where $N_T(q)$ is the number of zeros of

$$F(t) = x(t) - q \tag{11}$$

Analytic expression for $L(q)$ was first given by Kac,[14] and a simpler derivation was given by Mazur and Montroll:[9]

$$L(q) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta^{-2}\cos q\alpha \times\left[\prod_{k=0}^{N} J_0(a_k\alpha) - \prod_{k=0}^{N} J_0\left(a_k\left(\alpha^2+\eta^2\Omega_k^2\right)^{1/2}\right)\right]d\alpha\,d\eta \tag{12}$$

The third useful quantity for characterizing $x(t)$ is its "mean motion" or mean angular frequency defined as follows:

Let $x(t)$ or Eq. (6) be the real part of a complex quantity $z(t)$ given by

$$z(t) = \sum_{k=0}^{N} a_k e^{i(\Omega_k t+\delta_k)} \tag{13}$$

Write Eq. (13) as

$$z(t) = r(t)e^{i\phi(t)} \tag{14}$$

Then the mean motion or the mean angular frequency $\bar{\Omega}$ is defined by

$$\bar{\Omega} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \phi'(t)\,dt \tag{15}$$

Weyl[7] and Wintner[6] obtained the following expression for $\bar{\Omega}$ in terms of the weighted sum of Ω_k's:

$$\bar{\Omega} = \sum_{k=0}^{N} W_k\Omega_k \tag{16}$$

where W_k has the following interpretation in terms of Pearson's random walk: it is the probability that the random walker walking in a two-dimensional space, in a sequence of N steps of lengths $a_0, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_N$ spans a distance less than a_k. This probability is given by

$$W_k = a_k \int_0^{\infty} J_1(a_k\rho) \prod_{j=0,\, j\neq k}^{N} J_0(a_j\rho)\, d\rho \tag{17}$$

Using Eqs. (9), (12), (16), and (17), we now summarize our results[5] for the various quantities $P(x)$, $L(q)$, and $\overline{\Omega}$ which characterize the long time behavior of the atomic inversion $w(t)$ of our problem given by Eq. (2) when the initial photon distribution is (a) coherent and (b) chaotic. When the mean photon number $\bar{n}$ is $\gg 1$ (in practice $\bar{n} > 5$, say), we found that

$$P(x) \simeq \frac{1}{\sqrt{\pi}\,\sigma} \exp\left(-\frac{x^2}{\sigma^2}\right) \tag{18}$$

where

$$\sigma^2 \simeq \begin{cases} (1/4\pi\bar{n})^{1/2} & \text{for the initial Poissonian distribution} \quad (19a) \\ 1/2\bar{n} & \text{for the initial thermal distribution} \quad (19b) \end{cases}$$

$$L(q) \simeq \frac{\omega}{\pi\sigma} \operatorname{esp}\left(-\frac{q^2}{\sigma^2}\right) \tag{20}$$

where σ is given by Eqs. (19) and

$$\omega^2 \simeq \begin{cases} 2\sqrt{\bar{n}}\,\lambda^2/\sqrt{\pi} & \text{for the initial Poissonian distribution} \quad (21a) \\ \lambda^2 & \text{for the initial thermal distribution} \quad (21b) \end{cases}$$

and

$$\overline{\Omega} \simeq \begin{cases} 2\sqrt{\bar{n}}\,\lambda & \text{for the initial Poissonian distribution} \quad (22a) \\ \sqrt{\frac{\pi}{2}}\,\sqrt{\bar{n}}\,\lambda & \text{for the initial thermal distribution} \quad (22b) \end{cases}$$

The extent to which the quantities and expressions given by Eqs. (18)–(22) "characterize" the apparently "irregular" behavior of the atomic inversion $w(t)$ given by Eq. (2) can be better appreciated by the following observation: Figs. 2a, 2b and 2c showed the function $w(t)$ with different initial mean photon numbers corresponding to a thermal distribution. Our results given by Eqs. (18)–(22) predict that the record of $w(t)$ should be invariant to changes in $\bar{n}$ if $w(t)/\sigma$ is plotted instead of $w(t)$, and if the time unit is $\overline{\Omega}t$ instead of t. Figures 3a, 3b and 3c provide a test of this prediction. They showed the records of $w(t)/\sigma$ vs. $\overline{\Omega}t$ for the same three different values of $\bar{n}$ as in Figs. 2. The distributions are now seen to be

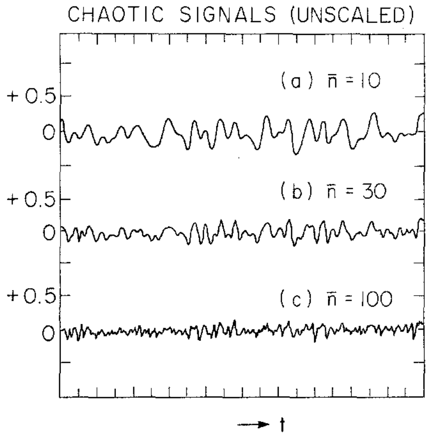

Fig. 2. Portions of long-time record of $w(t)$ with thermal p_k and $m = -1$ for (a) $\bar{n} = 10$, (b) $\bar{n} = 30$, and (c) $\bar{n} = 100$.

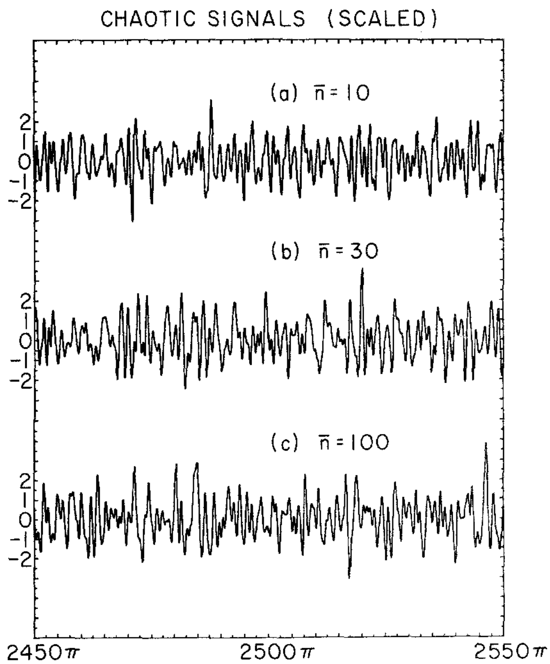

Fig. 3. Same portions of long-time record of $w(t)$ as in Fig. 2 but with the quantities now scaled according to $w(t)/\sigma$ and $\bar{\Omega}t$.

remarkably similar. This is another way of seeing that we have found the principal characteristics of the function $w(t)$.

From the mathematical point (as well as physical point) of view, the region of even greater interest is when $\bar{n}$ is small, say, $0 < \bar{n} < 5$. In this region, especially in the region $0 < \bar{n} < 1$, $P(x)$ and $L(q)$ are far from being Gaussian. Consider the exact expression of $P(x)$ given by Eq. (9). To be specific, let us take, in Eq. (2), $m = +1$ (i.e., the atom initially in the upper state), $p_k = (\bar{n})^k e^{-\bar{n}}/k!$ (i.e., the photon field to be initially coherent), and $\Delta \cong 0$. Then noting that in Eq. (9), for the case $\bar{n} = 0$,

$$a_k = p_k = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k = 1, 2, \ldots \end{cases} \tag{23}$$

and $J_0(0) = 1$, we find

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_0(\alpha) e^{-i\alpha x} \, d\alpha = \begin{cases} \dfrac{1}{\pi} \dfrac{1}{(1 - x^2)^{1/2}} & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases} \tag{24}$$

which is plotted in Fig. 4a. As $\bar{n}$ increases slightly from zero, we have

$$a_0 \lesssim 1, \qquad a_1, a_2, \ldots \cong 0$$

If we now assume that $a_2, a_3, \ldots$ can be set exactly equal to zero, then $P(x)$ becomes

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_0(a_0 \alpha) J_0(a_1 \alpha) e^{-i\alpha x} \, d\alpha \tag{25}$$

This integral can be evaluated exactly[15] and the result can be expressed in terms of the complete elliptic integral of the first kind. This is shown in Fig. 4b where $P(x)$ has logarithmic infinities at $x = \pm(a_0 - a_1)$. The inclusion of $J_0(a_2\alpha)J_0(a_3\alpha) \ldots$ in Eq. (9) with $N \to \infty$ would have the effect of "smoothing" out the function $P(x)$ to the point not only that it would become differentiable everywhere but also that it would possess derivatives of arbitrarily high order.[6] However, the smoothed curve is still very different from Gaussian.

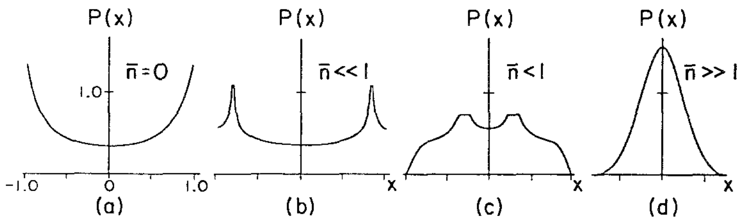

Fig. 4. Qualitative sketches of the probability density $P(x)$ for different values of $\bar{n}$.

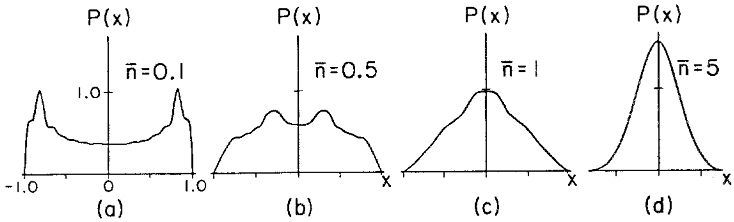

Fig. 5. Probability density $P(x)$ for different values of $\bar{n}$.

If $\bar{n}$ is increased further but still very much < 1, and if we continue the process of using only a finite number of $J_0(a_k\alpha)$ in the integrand of Eq. (9) and represent $P(x)$ now by

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} J_0(a_0\alpha)J_0(a_1\alpha)J_0(a_2\alpha)e^{-i\alpha x}\,d\alpha \tag{26}$$

then we have already come to the point when $P(x)$ can no longer be expressed in closed form. But expression such as (26) has been studied and numerically plotted in lattice dynamics[15] and this is shown qualitatively in Fig. 4c. Again the inclusion of the factors $J_0(a_3\alpha)J_0(a_4\alpha)\ldots$ in the integrand would have the effect of smoothing out the various sharp corners in Fig. 4c, but the curve for $P(x)$ is still far from being Gaussian.

As $\bar{n}$ increases further, we expect the distribution curve to become smoother, quickly approaching a Gaussian form as $\bar{n} > 5$, as shown in Fig. 4d.

The more precise probability density distributions obtained numerically when $J_0(a_2\alpha)J_0(a_3\alpha)\ldots$ is included are shown in Figs. 5(a)–(d).

It will be recognized that expressions given by Eqs. (24), (25), and (26) are precisely those representing the frequency distribution in lattice dynamics[15] in the harmonic approximation in one, two, and three dimensions, respectively. Thus increasing $\bar{n}$ from the value 0 in our problem has the same effect as increasing the dimensionality of the lattice considered in the study of its vibrational frequency spectrum. The lattice frequency spectrum in turn is closely related to the lattice random walk generating function.[15–19] It is interesting how the study of a physical observable of a simple quantum model (1) has brought together such a widely different collection of topics.

ACKNOWLEDGMENTS

I would like to thank Professor J. H. Eberly for first suggesting a statistical study of the Jaynes–Cummings model, and Professor Elliott

Montroll, from whose lectures given some years ago I learned of the beautiful work of A. Wintner, H. Weyl, M. Kac, as well as his own work in the related subject.

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